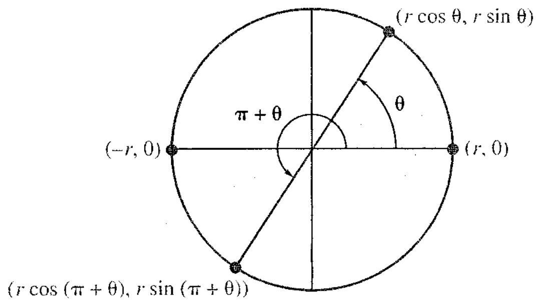


Figure 11

$$f(\theta) = T(r\cos\theta, r\sin\theta) - T(r\cos(\theta+\pi), r\sin(\theta+\pi))$$

With this definition

$$f(0) = T(r,0) - T(-r,0)$$
$$f(\pi) = T(-r,0) - T(r,0) = -[T(r,0) - T(-r,0)] = -f(0)$$

Thus, either $f(0)$ and $f(\pi)$ are both zero, or one is positive and the other is negative. If both are zero, then we have found the required two points. Otherwise, we can apply the Intermediate Value Theorem. Assuming that temperature varies continuously, we conclude that there exists a c between 0 and π such that $f(c) = 0$. Thus, for the two points at the angles c and $c + \pi$, the temperatures are the same. ■

Concepts Review

1. A function f is continuous at c if ________ $= f(c)$.

2. The function $f(x) = [\![x]\!]$ is discontinuous at ________.

3. A function f is said to be continuous on a closed interval $[a, b]$ if it is continuous at every point of (a, b) and if ________ and ________.

4. The Intermediate Value Theorem says that if a function f is continuous on $[a, b]$ and W is a number between $f(a)$ and $f(b)$, then there is a number c between ________ and ________ such that ________.

Problem Set 2.9

In Problems 1–15, state whether the indicated function is continuous at 3. If it is not continuous, tell why.

1. $f(x) = (x-3)(x-4)$

2. $g(x) = x^2 - 9$

3. $h(x) = \dfrac{3}{x-3}$

4. $g(t) = \sqrt{t-4}$

5. $h(t) = \dfrac{|t-3|}{t-3}$

6. $h(t) = \dfrac{|\sqrt{(t-3)^4}|}{t-3}$

7. $f(t) = |t|$

8. $g(t) = |t-2|$

9. $h(x) = \dfrac{x^2-9}{x-3}$

10. $f(x) = \dfrac{21-7x}{x-3}$

11. $r(t) = \begin{cases} \dfrac{t^3-27}{t-3} & \text{if } t \neq 3 \\ 27 & \text{if } t = 3 \end{cases}$

12. $r(t) = \begin{cases} \dfrac{t^3-27}{t-3} & \text{if } t \neq 3 \\ 23 & \text{if } t = 3 \end{cases}$

13. $f(t) = \begin{cases} t-3 & \text{if } t \le 3 \\ 3-t & \text{if } t > 3 \end{cases}$

14. $f(t) = \begin{cases} t^2-9 & \text{if } t \le 3 \\ (3-t)^2 & \text{if } t > 3 \end{cases}$

15. $f(x) = \begin{cases} -3x+7 & \text{if } x \le 3 \\ -2 & \text{if } x > 3 \end{cases}$

16. From the graph of g (see Figure 12), indicate the values where g is discontinuous. For each of the values state whether g is continuous from the right, left, or neither.

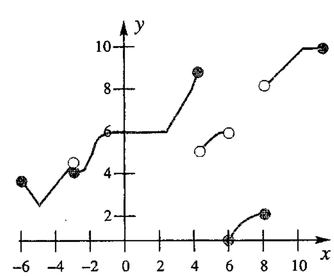


Figure 12

17. From the graph of h given in Figure 13, indicate the intervals on which h is continuous.

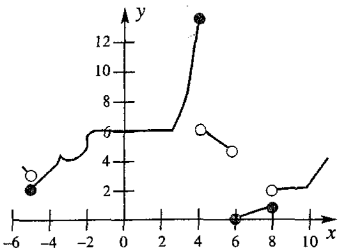


Figure 13

In Problems 18–23, the given function is not defined at a certain point. How should it be defined in order to make it continuous at this point? (See Example 1.)

18. $f(x) = \dfrac{x^2 - 49}{x - 7}$ **19.** $f(x) = \dfrac{2x^2 - 18}{3 - x}$

20. $g(\theta) = \dfrac{\sin(\theta)}{\theta}$ **21.** $H(t) = \dfrac{\sqrt{t} - 1}{t - 1}$

22. $\phi(x) = \dfrac{x^4 + 2x^2 - 3}{x + 1}$ **23.** $F(x) = \sin \dfrac{x^2 - 1}{x + 1}$

In Problems 24–35, at what points, if any, are the functions discontinuous?

24. $f(x) = \dfrac{3x + 7}{(x - 30)(x - \pi)}$

25. $f(x) = \dfrac{33 - x^2}{x\pi + 3x - 3\pi - x^2}$

26. $h(\theta) = |\sin\theta + \cos\theta|$ **27.** $r(\theta) = \tan\theta$

28. $f(u) = \dfrac{2u + 7}{\sqrt{u + 5}}$ **29.** $g(u) = \dfrac{u^2 + |u - 1|}{\sqrt[3]{u + 1}}$

30. $F(x) = \dfrac{1}{\sqrt{4 + x^2}}$ **31.** $G(x) = \dfrac{1}{\sqrt{4 - x^2}}$

32. $f(x) = \begin{cases} x & \text{if } x < 0 \\ x^2 & \text{if } 0 \le x \le 1 \\ 2 - x & \text{if } x > 1 \end{cases}$

33. $g(x) = \begin{cases} x^2 & \text{if } x < 0 \\ -x & \text{if } 0 \le x \le 1 \\ x & \text{if } x > 1 \end{cases}$

34. $f(t) = [\![t]\!]$ **35.** $g(t) = [\![t + \frac{1}{2}]\!]$

36. Sketch the graph of a function f that satisfies all the following conditions.

(a) Its domain is $[-2, 2]$.
(b) $f(-2) = f(-1) = f(1) = f(2) = 1$.
(c) It is discontinuous at -1 and 1.
(d) It is right continuous at -1 and left continuous at 1.

37. Let

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ -x & \text{if } x \text{ is irrational} \end{cases}$$

Sketch the graph of this function as best you can and decide where it is continuous.

38. Use the Intermediate Value Theorem to prove that $x^3 + 3x - 2 = 0$ has a real solution between 0 and 1.

39. Use the Intermediate Value Theorem to prove that $(\cos t)t^3 + 6\sin^5 t - 3 = 0$ has a real solution between 0 and 2π.

40. Show that the equation $x^5 + 4x^3 - 7x + 14 = 0$ has at least one real solution. *Hint*: Intermediate Value Theorem.

41. Prove that f is continuous at c if and only if $\lim_{t \to 0} f(t + c) = f(c)$.

42. Prove that if f is continuous at c and $f(c) > 0$ there is an interval $(c - \delta, c + \delta)$ such that $f(x) > 0$ on this interval.

43. Prove that, if f is continuous on $[0, 1]$ and satisfies $0 \le f(x) \le 1$ there, then f has a *fixed point*; that is, there is a number c in $[0, 1]$ such that $f(c) = c$. *Hint*: Apply the Intermediate Value Theorem to $g(x) = x - f(x)$.

44. Find the values of a and b so that the following function is continuous everywhere.

$$f(x) = \begin{cases} x + 1 & \text{if } x < 1 \\ ax + b & \text{if } 1 \le x < 2 \\ 3x & \text{if } x \ge 2 \end{cases}$$

45. A stretched elastic string covers the interval $[0, 1]$. The ends are released and the string contracts so that it covers the interval $[a, b]$, $a \ge 0$, $b \le 1$. Prove that this results in one point of the string (actually exactly one point) being where it was originally. See Problem 43.

46. Let $f(x) = \dfrac{1}{x - 1}$. Then $f(-2) = -\dfrac{1}{3}$ and $f(2) = 1$. Does the Intermediate Value Theorem imply the existence of a number c between -2 and 2 such that $f(c) = 0$? Explain.

47. Starting at 4 A.M., a hiker slowly climbed to the top of a mountain, arriving at noon. The next day, he returned along the same path, starting at 5 A.M. and getting to the bottom at 11 A.M. Show that at some point along the path his watch showed the same time on both days.

48. Let D be a bounded but otherwise arbitrary region in the first quadrant. Given an angle θ, $0 \le \theta \le \pi/2$, D can be circumscribed by a rectangle whose base makes angle θ with the x-axis as shown in Figure 14. Prove that at some angle this rectangle is a square. (This means that *any* bounded region can be circumscribed by a *square*.)

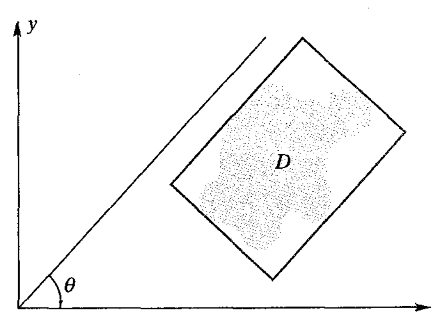


Figure 14

49. Let $f(x + y) = f(x) + f(y)$ for all x and y in $\mathbb{R}$ and suppose that f is continuous at $x = 0$.

(a) Prove that f is continuous everywhere.
(b) Prove that there is a constant m such that $f(t) = mt$ for all t in $\mathbb{R}$ (see Problem 41 of Section 2.1).

Problem Set 2.9

1. Continuous

3. Not continuous; $\lim_{x\to 3} \frac{3}{x-3}$ and $h(3)$ do not exist.

5. Not continuous; $\lim_{t\to 3} \frac{|t-3|}{t-3}$ and $h(3)$ do not exist.

7. Continuous **9.** Not continuous; $h(3)$ does not exist.

11. Continuous **13.** Continuous **15.** Continuous

17. $(-\infty, -\frac{5}{2}) \cup [-\frac{5}{2}, 4] \cup (4, 6) \cup [6, 8] \cup (8, \infty)$

19. Define $f(3) = -12$. **21.** Define $H(1) = \frac{1}{2}$.

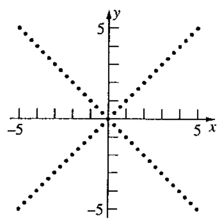
23. Define $F(-1) = -\sin 2$. **25.** $3, \pi$

27. Every $\theta = n\pi + \frac{\pi}{2}$ where n is any integer.

29. -1 **31.** $(-\infty, -2] \cup [2, \infty)$

33. 1 **35.** Every $t = n + \frac{1}{2}$ where n is any integer.

37.



Discontinuous everywhere except $x = 0$

57. (a)